Augmenting Paths

one more matched edge

—• = unmatched

∿∿ = matched

M = any matching $\bar{M}$ = max (card.) matching

$M \oplus \bar{M}$ = edges in exactly one of $M, \bar{M}$:

—• $\bar{M}$ subgraph, all degrees ≤ 2:

∿∿ M

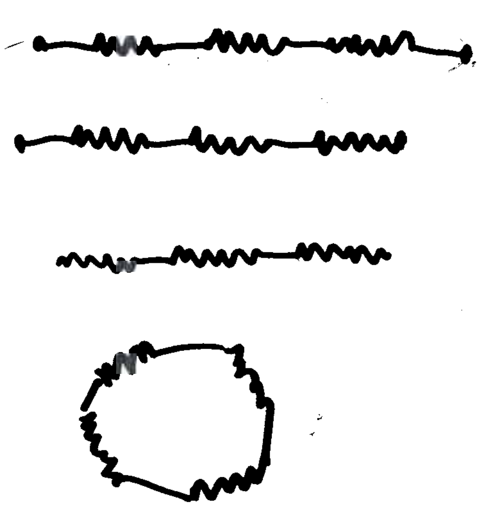
$\bar{M}$ excess

+1 M-augmenting

0

-1

0



If $|\bar{M}| - |M| = k$, $M \oplus \bar{M}$ contains k M-aug. paths

Max card matching

Begin with empty matching.

Repeatedly find an augmenting path, augment.

Stop when no more augmenting paths.

Bipartite case:

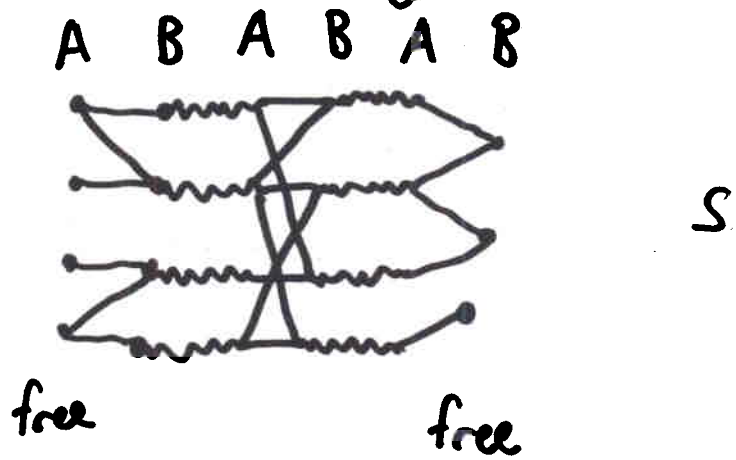
$O(m)$ time per augmentation.

$O(n)$ augmentations

$\Rightarrow O(nm)$ time total.

Bipartite case faster

Build layered subgraph containing all shortest aug paths by BFS



Find aug paths in S 1 at a time by DFS

Total time per phase $= O(m)$.

Length of shortest aug path strictly increases after a phase

$O(\sqrt{n})$ phases $\Rightarrow O(\sqrt{n}\, m)$ time

Each phase increases aug path length:

Let $d(v)$ be shortest dist from an A-free vertex to v via an alternating path.

$d(v)$'s strictly increase along any shortest aug. path. New edges created by a shortest aug. go from larger to smaller $d(v)$.

Thus no shorter aug path created by a shortest aug; after a phase, every aug path contains at least one edge from larger to smaller $d(v) \Rightarrow$ longer path.

$2\sqrt{n}$ phases:

Each phase increases matching size.

If $|\bar{M}| - |M| > \sqrt{n}$, $M \oplus \bar{M}$ contains $> \sqrt{n}$ aug paths, at least one of length $< \sqrt{n}$ (only n vertices).

$\Rightarrow$ After $\sqrt{n}$ phases, shortest aug path has length $\geq \sqrt{n}$ $\Rightarrow$ within $\sqrt{n}$ of max $\Rightarrow$ $\leq \sqrt{n}$ more phases.